STEP MATHEMATICS 2

2023

Mark Scheme

Question			Answer	Mark
1	(i)		$\int_a^b \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = \int_{a^{-1}}^{b^{-1}} \frac{1}{\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \cdot -\frac{1}{t^2}\,dt$	M1
			$= \int_{a^{-1}}^{b^{-1}} \frac{-t}{\left(1+t^2\right)^{\frac{3}{2}}}\,dt$	A1
				[2]
	(ii)	(a)	$\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = \int_{2}^{\frac{1}{2}} \frac{-t}{\left(1+t^2\right)^{\frac{3}{2}}}\,dt$	M1
			$= \left[(1+t^2)^{-\frac{1}{2}}\right]_2^{\frac{1}{2}}$	M1
			$= \left(\frac{5}{4}\right)^{-\frac{1}{2}} - 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$	A1
				[3]
		(b)	The integrand is even, so $\int_{-2}^{2} \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx = 2\int_0^2 \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx$	M1
			$= \lim_{\varepsilon\to 0} \int_{\varepsilon}^{2} \frac{2}{\left(1+x^2\right)^{\frac{3}{2}}}\,dx$ which is of the form in the stem	M1
			$= \lim_{\varepsilon\to 0} \left[2(1+t^2)^{-\frac{1}{2}}\right]_{\frac{1}{\varepsilon}}^{\frac{1}{2}}$	M1
			Dealing with limiting process for the lower limit	E1
			$= \frac{4}{\sqrt{5}}$	A1
			$-\lim_{\varepsilon\to 0} \frac{2\varepsilon}{\sqrt{1+\varepsilon^2}}$	A1
			so the limit is zero and the integral $= \frac{4}{\sqrt{5}}$	E1
				[7]
	(iii)	(a)	$\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^{2}}\,dx = \int_{2}^{\frac{1}{2}} \frac{-t^2}{\left(1+t^2\right)^{2}}\,dt$	M1
			$= \int_{\frac{1}{2}}^{2} \frac{x^2}{\left(1+x^2\right)^{2}}\,dx$	A1

			so $\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{\left(1+x^2\right)^2}\,dx + \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{x^2}{\left(1+x^2\right)^2}\,dx$	**M1**
			$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,dx$	**A1**
			$= \frac{1}{2}\left(arctan\,2 - arctan\frac{1}{2}\right)$	**B1**
				[5]
		(b)	$\int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}}\,dx = \int_{2}^{\frac{1}{2}} \frac{t-1}{\left(1+\frac{1}{t^2}\right)^{\frac{1}{2}}} \cdot -\frac{1}{t^2}\,dt$	**M1**
			$= \int_{2}^{\frac{1}{2}} \frac{1-t}{t\left(1+t^2\right)^{\frac{1}{2}}}\,dt$	**A1**
			$= -\int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}}\,dx$, so the integral = 0.	**E1**
				[3]

Question			Answer	Mark
2	(i)		Let $x = \tan\alpha$. Then $y = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$	
			so $z = \tan 4\alpha$, and so $\tan\alpha = \tan 8\alpha$.	E1
			giving $8\alpha = \alpha + n\pi$, or $\alpha = \frac{1}{7}n\pi$, (for n = –3 to 3).	M1
			Solutions are: (0,0,0), $\left(\tan\left(\frac{1}{7}\alpha\right), \tan\left(\frac{2}{7}\alpha\right), \tan\left(-\frac{3}{7}\alpha\right)\right)$	B1
			and cyclic permutations of the latter	A1
			and $\left(\tan\left(-\frac{1}{7}\alpha\right), \tan\left(-\frac{2}{7}\alpha\right), \tan\left(\frac{3}{7}\alpha\right)\right)$ and its cyclic permutations	A1
				[5]
	(ii)		$\tan 3\alpha = \frac{\frac{2\tan\alpha}{1-\tan^2\alpha}+\tan\alpha}{1-\frac{2\tan^2\alpha}{1-\tan^2\alpha}}$	M1
			$= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$	A1
			Let $x = \tan\alpha$; then $y = \tan 3\alpha$, $z = \tan 9\alpha$, so $\tan 27\alpha = \tan\alpha$	M1
			giving $26\alpha = n\pi$	A1
			which has 25 solutions with distinct values of $\tan\alpha$ because n = 13 does not give a possible value of $\tan\alpha$.	A1
			Checking that for each finite value of x, the denominators of y and z are defined (i.e. checking 1-3x^2 is non-zero).	E1
				[6]
	(iii)	(a)	Let $x = \cos\alpha$	M1
			the restriction on $\lvert x\rvert$ means this is a complete parametrisation of solutions	E1
			Then, using $\cos 2\alpha = 2\cos^2\alpha - 1$, $\cos 8\alpha = \cos\alpha$	M1
			so $8\alpha = \alpha + 2m\pi$, or $8\alpha = -\alpha + 2n\pi$	M1
			so $7\alpha = 2m\pi$ or $9\alpha = 2n\pi$	A1
			with 4 (m = 0 to 3) + 5 (n = 0 to 4)	M1
			– 1 (for $\alpha = 0$ twice) = 8 distinct solutions	A1
				[7]
		(b)	y quadratic, so z quartic in x, so x satisfies an octic equation	B1
			which has at most 8 roots, so there are no larger solutions.	E1
				[2]

Question			Answer	Mark
3	(i)	(a)	An odd degree polynomial takes positive and negative values for large enough $\|x\|$.	B1
			The degree of q is equal to the degree of p, and the coefficient of x^n is positive, because each derivative has lower degree and cannot affect the coefficient of x^n.	M1
			So $q(x) > 0$ for large enough $\|x\|$.	A1
				[3]
		(b)	$q'(x) = \sum_{k=0}^{n} p^{(k+1)}(x) = \sum_{k=0}^{n-1} p^{(k+1)}(x)$ $[p^{(n+1)}(x) \equiv 0]$	M1
			$= \sum_{k=1}^{n} p^{(k)}(x) = q(x) - p(x)$	A1
				[2]
	(ii)	(a)	If $q'(x) = 0$, $q(x) = p(x)$, so the two curves meet at any stationary point.	B1
			But $q(x) > 0$ for large enough $\|x\|$.	M1
			So $q(x)$ has an absolute minimum value	M1
			at which its value is positive, as $q(x) = p(x)$ there.	A1
				[4]
		(b)	$\frac{d}{dx}(e^{-x} q(x)) = e^{-x}\left(-q(x) + q'(x)\right)$	M1
			$= -e^{-x} p(x) < 0$	A1
			For large enough x, $q(x) > 0$, so $e^{-x} q(x) > 0$,	M1
			but $e^{-x} q(x)$ decreasing, so positive for all x,	A1
			and hence so is $q(x)$.	A1
				[5]
		(c)	$\int_0^\infty p(x+t)e^{-t}\,dt$ $= \left[-p(x+t)e^{-t}\right]_0^\infty + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$	M1
			$= \left(0 - (-p(x))\right) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$	A1
			So $\int_0^\infty p(x+t)e^{-t}\,dt = p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,dt$ $= p(x) + p^{(1)}(x) + \int_0^\infty p^{(2)}(x+t)e^{-t}\,dt$	M1
			$= p(x) + p^{(1)}(x) + \ldots + p^{(n)}(x) + \int_0^\infty p^{(n+1)}(x+t)e^{-t}\,dt$	M1
			but $p^{(n+1)}(x) \equiv 0$, so $\int_0^\infty p(x+t)e^{-t}\,dt = p(x) + p^{(1)}(x) + \cdots + p^{(n)}(x) = q(x)$.	A1
			but $p(x+t)$, $e^{-t} > 0$ for all $t \geqslant 0$, so $q(x) > 0$.	E1
				[6]

Question			Answer	Mark
4	(i)		$\left(x-\sqrt{2}\right)^2 = 3 \Rightarrow x^2 - 2\sqrt{2}x - 1 = 0$	M1
			so $\left(x^2 - 2\sqrt{2}x - 1\right)\left(x^2 + 2\sqrt{2}x - 1\right) = 0$	M1
			that is, $x^4 - 10x^2 + 1 = 0$	A1
			but $\sqrt{2}+\sqrt{3}$ a root of $\left(x-\sqrt{2}\right)^2 = 3$ so a root of $x^4 - 10x^2 + 1 = 0$.	A1
				[4]
	(ii)		$\left(\sqrt{3}+\sqrt{5}\right)^2 = 8 + 2\sqrt{15}$	B1
			$\sqrt{2}+\sqrt{3}+\sqrt{5}$ a root of $\left(x-\sqrt{2}\right)^2 - \left(8+2\sqrt{15}\right)$	M1
			that is, of $x^2 - 2\sqrt{2}x - 6 - 2\sqrt{15}$	A1
			so of $\left(x^2 - 2\sqrt{2}x - 6 - 2\sqrt{15}\right)\left(x^2 + 2\sqrt{2}x - 6 + 2\sqrt{15}\right)$	M1
			= $x^4 - 20x^2 - 8\sqrt{30}x - 24$	A1
			so of $\left(x^4 - 20x^2 - 8\sqrt{30}x - 24\right)\left(x^4 - 20x^2 + 8\sqrt{30}x - 24\right)$	M1
			= $x^8 - 40x^6 + 352x^4 - 960x^2 + 576$	A1
				[7]
Alternative				
	(ii)		The roots will be the eight numbers of the form $\pm\sqrt{2}\pm\sqrt{3}\pm\sqrt{5}$	B1
			Which can be paired as $\pm\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)$, $\pm\left(\sqrt{2}-\sqrt{3}-\sqrt{5}\right)$, $\pm\left(-\sqrt{2}+\sqrt{3}-\sqrt{5}\right)$, $\pm\left(-\sqrt{2}-\sqrt{3}+\sqrt{5}\right)$	
			So the polynomial is a quartic in x^2 with roots $\alpha = \left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)^2$, $\beta = \left(\sqrt{2}-\sqrt{3}-\sqrt{5}\right)^2$, $\gamma = \left(-\sqrt{2}+\sqrt{3}-\sqrt{5}\right)^2$, $\delta = \left(-\sqrt{2}-\sqrt{3}+\sqrt{5}\right)^2$	
			$\alpha+\beta+\gamma+\delta = 4(2+3+5) = 40$	M1
			$\alpha\beta\gamma\delta = \left(2-\left(\sqrt{3}+\sqrt{5}\right)^2\right)^2\left(2-\left(\sqrt{3}-\sqrt{5}\right)^2\right)^2$ $= \left(\left(-6-2\sqrt{15}\right)\left(-6+2\sqrt{15}\right)\right)^2 = (36-60)^2 = 576$	A1
			$\alpha^2 = \left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)^4$ $= 4+9+25+4\left(5\sqrt{6}+7\sqrt{10}+8\sqrt{15}\right)$ $+6(6+10+15)$ $= 224 + 4\left(5\sqrt{6}+7\sqrt{10}+8\sqrt{15}\right)$	
			So $\alpha^2+\beta^2+\gamma^2+\delta^2 = 4\times 224 = 896$	
			$2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) =$ $(\alpha+\beta+\gamma+\delta)^2 - (\alpha^2+\beta^2+\gamma^2+\delta^2)$	M1
			$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{40^2-896}{2} = 352$	A1
			$\alpha^3+\beta^3+\gamma^3+\delta^3 =$ $4(8+27+125+15(12+20+18+45+50+75)+90(30))$ $= 24640$	

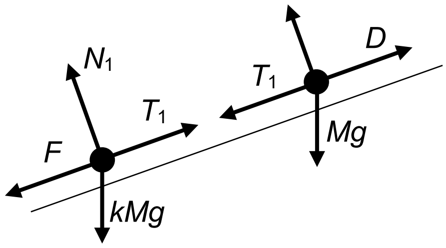
			$(\alpha+\beta+\gamma+\delta)^3=(\alpha^3+\beta^3+\gamma^3+\delta^3)$ $+3(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\alpha+\beta+\gamma+\delta)$ $-3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)$	**M1**
			$3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)=24640+3(352)(40)-40^3$	
			$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=\frac{2880}{3}=960$	
			Therefore the polynomial is $x^8-40x^6+352x^4-960x^2+576$	**A1**
				[7]
	(iii)		$a+\sqrt{2}$, $b+\sqrt{2}$, $c+\sqrt{2}$ are roots of $\left(x-\sqrt{2}\right)^3-3\left(x-\sqrt{2}\right)+1$	**M1**
			so of $x^3-3\sqrt{2}x^2+3x+\sqrt{2}+1$	**A1**
			so of $\left(x^3-3\sqrt{2}x^2+3x+\sqrt{2}+1\right)\left(x^3+3\sqrt{2}x^2+3x-\sqrt{2}+1\right)$	**M1**
			= $x^6-12x^4+2x^3+21x^2+6x-1$	**A1**
				[4]
	(iv)		$\left(\sqrt[3]{2}+\sqrt[3]{3}\right)^3=5+3\sqrt[3]{12}+3\sqrt[3]{18}$	**M1**
			$=5+3\sqrt[3]{6}\left(\sqrt[3]{2}+\sqrt[3]{3}\right)$	**A1**
			so $\sqrt[3]{2}+\sqrt[3]{3}$ satisfies $x^3-5=3\sqrt[3]{6}x$	**M1**
			so satisfies $(x^3-5)^3=162x^3$	**M1**
			or $x^9-15x^6-87x^3-125=0$	**A1**
				[5]

Question			Answer	Mark
5	**(i)**	**(a)**	$x_{n+1} - 1 = \frac{1}{x_n+1}$; $x_0 \geqslant 1$	**M1**
			so if $x_n \geqslant 1$, $x_{n+1} \geqslant 1$	**A1**
				[2]
		(b)	$x_{n+1}^2 - 2 = \frac{(x_n+2)^2 - 2(x_n+1)^2}{(x_n+1)^2} = -\frac{x_n^2 - 2}{(x_n+1)^2}$	**M1**
			so $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, as $(x_n+1)^2 \geqslant 2^2 > 0$	**A1**
			$\lvert x_{n+1}^2 - 2 \rvert \leqslant \frac{1}{4} \lvert x_n^2 - 2 \rvert$, as $(x_n+1)^2 \geqslant 2^2$	**A1**
				[3]
		(c)	10 is even, so $x_{10}^2 - 2$ and $x_0^2 - 2$ have the same sign, which is negative, so $x_{10}^2 < 2$	**B1**
			and $\lvert x_{10}^2 - 2 \rvert \leqslant \frac{1}{4^{10}} \lvert x_0^2 - 2 \rvert = \frac{1}{4^{10}}$	**M1**
			$< 10^{-6}$, as $2^{10} > 10^3$	**A1**
			so $10^{-6} \geqslant 2 - x_{10}^2$ giving stated result	**A1**
				[4]
	(ii)	**(a)**	$y_{n+1} - \sqrt{2} = \frac{y_n^2 + 2 - 2\sqrt{2}y_n}{2y_n} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$	**B1**
			so $y_0 \geqslant 1$ and $y_{n+1} \geqslant \sqrt{2} \geqslant 1$ for $n \geqslant 0$	**B1**
				[2]
		(b)	$y_1 - \sqrt{2} = \frac{\left(1-\sqrt{2}\right)^2}{2} = 2\left(\frac{\sqrt{2}-1}{2}\right)^2$, so result holds for $n = 1$.	**B1**
			also $y_{n+1} - \sqrt{2} = \frac{\left(\lvert y_n - \sqrt{2} \rvert\right)^2}{2y_n} \leqslant \frac{2}{y_n}\left(\frac{\sqrt{2}-1}{2}\right)^{2^{n+1}}$	**M1**
			$\leqslant 2\left(\frac{\sqrt{2}-1}{2}\right)^{2^{n+1}}$, as $y_n \geqslant \sqrt{2}$ for $n \geqslant 1$	**A1**
			appropriate induction structure	**A1**
				[4]
		(c)	$y_{10} \geqslant \sqrt{2}$	**B1**
			$y_{10} - \sqrt{2} \leqslant 2\left(\frac{\sqrt{2}-1}{2}\right)^{2^{10}}$	**M1**
			but $\left(\frac{\sqrt{2}-1}{2}\right)^5 \leqslant \left(\frac{1}{4}\right)^5 \leqslant 10^{-3}$ $\left(\frac{\sqrt{2}-1}{2}\right)^5 \leqslant \left(\frac{1}{4}\right)^5 \leqslant 10^{-3}$	**M1**
			$y_{10} - \sqrt{2} \leqslant 2(10^{-3})^{204}$	**A1**
			$= 2 \times 10^{-612} < 10^{-600}$	**A1**
				[5]

Question			Answer	Mark
6			Induction structure	M1
			Base case	B1
			$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\F_n & F_{n-1}\end{pmatrix}=\begin{pmatrix}F_{n+1}+F_n & F_n+F_{n-1}\\F_{n+1} & F_n\end{pmatrix}$ or $\begin{pmatrix}F_{n+1} & F_n\\F_n & F_{n-1}\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}=\begin{pmatrix}F_{n+1}+F_n & F_{n+1}\\F_n+F_{n-1} & F_n\end{pmatrix}$	A1
			Use of definition (of F_n) and conclusion (of induction)	A1
				[4]
	(i)		Use of det(Q^n) = (det Q)^n	M1
			clearly shown	A1
				[2]
	(ii)		Use of (1,2) entry in $Q^{m+n} = Q^m Q^n$	M1
			clearly shown	A1
				[2]
	(iii)		$Q^2 = Q + I$	B1
				[1]
		(a)	Use of $Q^{2n} = (Q + I)^n$	M1
			and Binomial expansion	M1
			clearly shown	A1
				[3]
		(b)	Derivation of $Q^3 = Q(Q + I) = 2Q + I$ *(give the mark for any one of these)*	B1
			Use of $Q^{3n} = (2Q + I)^n$ and Binomial expansion	M1
			clearly shown	A1
			Use of $Q^{3n} = Q^n(Q + I)^n$ and Binomial expansion	M1
			clearly shown	A1
				[5]
		(c)	Use of $I=Q^n(Q - I)^n$ (or (-Q)^n (I-Q)^n)	M1
			Use of binomial expansion	M1
			clearly shown	A1
				[3]

Question			Answer	Mark
7	(i)		$\lvert zw\rvert^2 = \lvert(ac-bd)+i(ad+bc)\rvert^2$ $= (ac-bd)^2 + (ad+bc)^2$ $= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$	M1
			$\lvert z\rvert^2\lvert w\rvert^2 = (a^2+b^2)(c^2+d^2)$ $= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$	M1
			Therefore $\lvert zw\rvert^2 = \lvert z\rvert^2\lvert w\rvert^2$	A1
				[3]
	(ii)		$\lvert 2+i\rvert = \sqrt{5}$ and $\lvert 10+11i\rvert = \sqrt{221}$	B1
			so $9^2 + 32^2 = (2^2 + 1^2)(10^2 + 11^2) = 5 \times 221$	B1
				[2]
	(iii)		$8045 = 5 \times 1609$	M1
			$= (2^2 + 1^2)(40^2 + 3^2)$	M1
			so $\lvert(2 + i)(40 + 3i)\rvert^2 = 77^2 + 46^2 = 8045$ (also $34^2 + 83^2$)	A1
				[3]
	(iv)		$612^2 + 1206^2 = 6^2 \times 50805$	B1
				[1]
	(v)		$1002082 = 1001^2 + 9^2$	B1
			so one pair is $5005^2 + 45^2$	A1
			but $25 = 3^2 + 4^2$ and $(3 + 4i)(1001 + 9i)$ $= 2967 + 4031i$	M1
			so a second pair is $2967^2 + 4031^2$	A1
			also, $(4 + 3i)(1001 + 9i) = 3977 + 3039i$	M1
			so a third pair is $3977^2 + 3039^2$	A1
				[6]
	(vi)		require $(10^2 + 3^2)(c^2 + d^2) = (1001^2 + 6^2)$	M1
			implies simultaneous equations for *c* and *d*	M1
			$10c - 3d = 1001$, $10d + 3c = 6$ or $3c - 10d = 1001$, $3d + 10c = 6$	A1
			giving $c = 92$, $d = -27$ (from the first set)	A1
			so $9193 = 92^2 + 27^2$ (also $38^2 + 211^2$)	A1
				[5]

Question			Answer	Mark
8	(i)		Let vertices be numbered 1 to 4 and edges be e_{ij}, where $i < j$. Then perimeters equal is $\lvert e_{12}\rvert + \lvert e_{23}\rvert + \lvert e_{13}\rvert = \lvert e_{12}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{13}\rvert + \lvert e_{34}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{24}\rvert + \lvert e_{34}\rvert + \lvert e_{23}\rvert$	M1
			which implies $\lvert e_{12}\rvert + \lvert e_{23}\rvert + \lvert e_{13}\rvert + \lvert e_{12}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert$ $= \lvert e_{13}\rvert + \lvert e_{34}\rvert + \lvert e_{14}\rvert + \lvert e_{24}\rvert + \lvert e_{34}\rvert + \lvert e_{23}\rvert$ so $2\lvert e_{12}\rvert + (\lvert e_{23}\rvert + \lvert e_{13}\rvert + \lvert e_{24}\rvert + \lvert e_{14}\rvert)$ $= 2\lvert e_{34}\rvert + (\lvert e_{13}\rvert + \lvert e_{14}\rvert + \lvert e_{24}\rvert + \lvert e_{23}\rvert)$ and so $\lvert e_{12}\rvert = \lvert e_{34}\rvert$	A1
			and by permutations of this argument, all pairs of opposite sides are equal.	E1
			if $\lvert e_{12}\rvert = \lvert e_{34}\rvert$, $\lvert e_{13}\rvert = \lvert e_{24}\rvert$, $\lvert e_{14}\rvert = \lvert e_{23}\rvert$, then all perimeters are trivially equal	B1
				[4]
	(ii)		$\lvert\boldsymbol{a}\rvert^2 = \lvert\boldsymbol{b} - \boldsymbol{c}\rvert^2$	M1
			$= \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\boldsymbol{b}.\boldsymbol{c}$	A1
			From the equivalent results to **(ii)** using the other pairs of opposite sides	M1
			$\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c} = \frac{1}{2}(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 - \lvert\boldsymbol{c}\rvert^2) + \frac{1}{2}(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{b}\rvert^2) = \lvert\boldsymbol{a}\rvert^2$	A1
				[4]
	(iii)		$16\lvert\boldsymbol{a} - \boldsymbol{g}\rvert^2 = \lvert 3\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2$	M1
			$= 9\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 6\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	
			$= 9\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 6\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$ using previous results	M1
			$= 2(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2)$	A1
			but this is symmetric in **a**, **b**, **c** so **g** equidistant from A, B and C.	A1
			$16\lvert\boldsymbol{g}\rvert^2 = \lvert\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 + 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + \lvert\boldsymbol{b} + \boldsymbol{c}\rvert^2$ $= \lvert\boldsymbol{a}\rvert^2 + 2\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 + 2\boldsymbol{b}.\boldsymbol{c}$ $= 3\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$ $= 2(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2)$ So G equidistant from O also.	B1
				[5]
	(iv)		$\lvert\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2 = \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) + 2\boldsymbol{b}.\boldsymbol{c}$	M1
			$= \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - 2\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2$	M1
			$= 2(\lvert\boldsymbol{b}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 - \lvert\boldsymbol{a}\rvert^2)$	A1
			which must be non-negative, so cos(BAC) $\geqslant 0$	M1
			and symmetry implies no angle obtuse	A1
			If e.g. BAC was a right angle, would have $\lvert\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}\rvert^2 = 0$, so **a** = **b** + **c**	M1
			so O, A, B, C all in one plane, so not a tetrahedron.	A1
				[7]

Question			Answer	Mark
9				B1
			$D - T_1 - Mg\sin\alpha = Ma_1$ **1** $T_1 - F - kMg\sin\alpha = kMa_1$ **2** $[D - \mu kMg\cos\alpha - (k+1)Mg\sin\alpha = (k+1)Ma_1]$	B1
			$N_1 - kMg\cos\alpha = 0$, so $F = \mu kMg\cos\alpha$ **3**	B1
			$D - T_3 + Mg\sin\alpha = Ma_3$ **4** $T_3 - \mu kMg\cos\alpha + kMg\sin\alpha = kMa_3$ **5** $[D - \mu kMg\cos\alpha + (k+1)Mg\sin\alpha = (k+1)Ma_3]$	B1
			$D - T_2 = Ma_2$ **6** $T_2 - \mu kMg = kMa_2$ **7** $[D - \mu kMg = (k+1)Ma_2]$	B1
				[5]
	(i)		**2** – k**1** gives $(1+k)T_1 = \mu kMg\cos\alpha + kD$	M1
			and **5** – k**4** gives $(1+k)T_3 = \mu kMg\cos\alpha + kD$ so $T_1 = T_3$.	A1
			1 + **4** – 2**6** gives $-T_1 - T_3 + 2T_2 = Ma_3 + Ma_1 - 2Ma_2$	M1
			which is the given result, using $T_1 = T_3$	A1
				[4]
	(ii)	(a)	**7** – k**6** gives $(1+k)T_2 = \mu kMg + kD$	M1
			so $T_2 > T_1$, as $\cos\alpha < 1$.	A1
			hence $a_1 + a_3 > 2a_2$,which is the middle inequality	A1
			also, $kMa_1 = T_1 - \mu kMg\cos\alpha - kMg\sin\alpha$ and $kMa_3 = T_3 - \mu kMg\cos\alpha + kMg\sin\alpha$	M1
			so $a_3 > a_1$, which also implies $a_3 > \frac{1}{2}(a_1 + a_3) > a_1$,	A1
				[5]
		(b)	(**7** + **6**) – (**2** + **1**) gives	M1
			$(1+k)(a_2 - a_1) = g\big((1+k)\sin\alpha - k\mu(1-\cos\alpha)\big)$	A1
			$= 2g\sin\left(\frac{1}{2}\alpha\right)\left((1+k)\cos\frac{1}{2}\alpha - k\mu\sin\frac{1}{2}\alpha\right)$	M1
			but $1 + k > \mu k$, as $\mu < 1$, and	M1
			$\cos\frac{1}{2}\alpha > \sin\frac{1}{2}\alpha$, as $\frac{1}{2}\alpha < 45°$	M1
			so $a_2 > a_1$, as required.	A1

				[6]

Question			**Answer**	**Mark**
10	**(i)**		$z = ut\sin\alpha - \frac{1}{2}gt^2$ and $x = ut\cos\alpha$	**M1**
			require $ut\sin\alpha - \frac{1}{2}gt^2 = A - B(ut\cos\alpha)^2$ to have a double root	**M1**
			so $\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right)t^2 + (u\sin\alpha)t - A = 0$ has zero discriminant	**A1**
			so $u^2\sin^2\alpha + 4A\left(Bu^2\cos^2\alpha - \frac{1}{2}g\right) = 0$	**A1**
			using $\sin^2\alpha = 1 - \cos^2\alpha$, gives $$u^2 - u^2\cos^2\alpha + 4ABu^2\cos^2\alpha - 2Ag = 0$$	**A1**
			If this is true for all α, both the left-hand side, and the coefficient of $\cos^2\alpha$ on the right-hand side must be zero	**M1**
			So $A = \frac{u^2}{2g}$ and $B = \frac{g}{2u^2}$	**A1**
				[7]
	(ii)		The set of points inside and on S are vulnerable	**M1**
			because if a point outside S was on a trajectory, it would have to cross S, which contradicts the fact that it touches to the surface.	**A1**
				[2]
	(iii)		Particles in the *x*-*z* plane have, eliminating α,	**M1**
			$$\left(z + \frac{1}{2}gt^2\right)^2 + x^2 = u^2t^2$$	**A1**
			so, rotating this circle about the *z*-axis, all the particles lie on a sphere	**M1**
			of radius *ut* and centre $\left(0,\ 0,\ -\frac{1}{2}gt^2\right)$	**A1**
				[4]
	(iv)		for *Q*, $z = \frac{u^2}{2g} - \frac{1}{2}gt^2$ and $x = ut$	**M1**
			so $z = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$, which is the equation of *E*	**A1**
				[2]
	(v)		Using **(i)**, P_α meets *E* at time *t* given by	**M1**
			$$g^2\sin^2\alpha\, t^2 - 2gu\sin\alpha\, t + u^2 = 0$$	**A1**
			so at $t = \frac{u}{g\sin\alpha}$	**A1**
			so require $u(t - T) = u\cos\alpha\, t$	**M1**
			that is $T = \frac{u}{g}\frac{1-\cos\alpha}{\sin\alpha}$	**A1**
				[5]

Question			Answer	Mark
11	(i)		E[Y] = $\sum_{r=1}^{n} x_r(pa_r + qb_r) = p\sum_{r=1}^{n} x_r a_r + q\sum_{r=1}^{n} x_r b_r = p\mu_1 + q\mu_2$	B1
			Var[Y] = $\sum_{r=1}^{n} x^2{}_r(pa_r + qb_r) - E[Y]^2$ = $p({\sigma_1}^2 + {\mu_1}^2) + q({\sigma_2}^2 + {\mu_2}^2) - (p\mu_1 + q\mu_2)^2$	M1
			$= p{\sigma_1}^2 + q{\sigma_2}^2 + (p - p^2){\mu_1}^2 + 2pq\mu_1\mu_2 + (q - q^2){\mu_2}^2$	M1
			giving the required result	A1
				[4]
	(ii)		B = 1 with probability $\frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{5}{6} = \frac{1}{2}$ and Z_1 counts the number of times out of n that B = 1	B1
			mean of Z_1 = $\frac{1}{2}n$ and variance of Z_1 = $\frac{1}{4}n$	B1
			so $P\left(\frac{1}{2}n - \frac{1}{20}n \leqslant Z_1 \leqslant \frac{1}{2}n + \frac{1}{20}n\right)$	M1
			$\approx P\left(-\frac{\frac{1}{20}n}{\frac{1}{2}\sqrt{n}} \leqslant S \leqslant \frac{\frac{1}{20}n}{\frac{1}{2}\sqrt{n}}\right)$ where S is standard Normal	A1
			but $\sqrt{n}$ gets large as n gets large/n grows faster than $\sqrt{n}$ as n gets large	M1
			and hence the probability tends to $P(-\infty < S < \infty) = 1$ 1 as $n \to \infty$	A1
				[6]
	(iii)		mean of Z_2 = $\frac{1}{2}\cdot\frac{1}{6}n + \frac{1}{2}\cdot\frac{5}{6}n = \frac{1}{2}n$	B1
			and variance of Z_2 = $\frac{1}{2}\cdot\frac{5}{36}n + \frac{1}{2}\cdot\frac{5}{36}n + \frac{1}{4}\left(\frac{1}{6}n - \frac{5}{6}n\right)^2$ = $\frac{5}{36}n + \frac{1}{9}n^2$	B1
			A Normal distribution with this mean and variance will not be a good approximation to the distribution of Z_2 because Z_2 is bimodal: it is likely to take values close to $\frac{1}{6}n$ or $\frac{5}{6}n$, not near $\frac{1}{2}n$	B1
			Z_2 is within 10% of its mean only if the coin shows heads and a surprisingly large number of sixes appear, or the coin shows tails, and surprisingly few sixes appear.	M1
			In the first case, the probability that a surprisingly large number of heads appears is less than	M1
			$P\left(B_1 \geqslant \frac{1}{2}n - \frac{1}{20}n\right)$ where B_1 is a Binomial variable with mean $\frac{1}{6}n$ and variance $\frac{5}{36}n$.	A1
			which, by Normal approximation, $\approx P\left(S \geqslant \frac{\frac{1}{2}n - \frac{1}{20}n - \frac{1}{6}n}{\frac{1}{6}\sqrt{5n}}\right)$	M1
			= $P\left(S \geqslant \frac{17\sqrt{n}}{10\sqrt{5}}\right)$	A1

			This is also greater than the probability of surprisingly few heads in the second case so also the probability that Z_2 is within 10% of its mean	**M1**
			and tends to 0 as $n \to \infty$	**A1**
				[10]

Question			Answer	Mark
12	(i)		$P(Y \leqslant t) = P(X_i \leqslant t \text{ for all } i = 1, \ldots, n)$ $= \prod_{i=1}^{n} P(X_i \leqslant t) = [P(X_1 \leqslant t)]^n$	E1
			$= \left(\int_0^t \frac{1}{2}\sin x \, dx\right)^n = \frac{1}{2^n}(1-\cos t)^n$	M1
			so $f_Y(t) = \frac{n \sin t}{2^n}(1-\cos t)^{n-1}$	A1
				[3]
	(ii)		$\frac{1}{2^n}(1-\cos m(n))^n = \frac{1}{2}$	M1
			so $m(n) = \arccos\left(1 - 2^{\frac{n-1}{n}}\right)$	A1
			which tends to π as $n \to \infty$.	A1
				[3]
	(iii)		$\mu(n) = \int_0^{\pi} x \frac{n}{2^n} \sin x (1-\cos x)^{n-1} \, dx$ $= \left[x \frac{1}{2^n}(1-\cos x)^n\right]_0^{\pi} - \int_0^{\pi} \frac{1}{2^n}(1-\cos x)^n \, dx$	M1
			$= \left(\pi \frac{1}{2^n} 2^n - 0\right) - \int_0^{\pi} \frac{1}{2^n}(1-\cos x)^n \, dx$ as required	A1
				[2]
		(a)	As $\mu(n) = \pi - \int_0^{\pi} \left(\frac{1-\cos x}{2}\right)^n dx$, the integrand decreases with n throughout $(0, \pi)$	M1
			and so $\mu(n)$ increases with n	A1
				[2]
		(b)	$\mu(2) = \pi - \int_0^{\pi} \frac{1}{4}\left(1 - 2\cos x + \cos^2 x\right) dx$ $= \pi - \int_0^{\pi} \frac{1}{4}\left(1 - 2\cos x + \frac{1}{2}(1+\cos 2x)\right) dx$	M1
			$= \pi - \left[\frac{3}{8}x - \frac{1}{2}\sin x + \frac{1}{16}\sin 2x\right]_0^{\pi}$	M1
			$= \frac{5}{8}\pi$	A1
			so $\cos^2(\mu(2)) = \frac{1}{2}\left(1 + \cos\frac{5}{4}\pi\right) = \frac{1}{4}\left(2 - \sqrt{2}\right)$	M1
			but $\cos^2(m(2)) = \left(1 - \sqrt{2}\right)^2 = 3 - 2\sqrt{2}$	M1
			which is greater, as $\left(3 - 2\sqrt{2}\right) - \frac{1}{4}\left(2 - \sqrt{2}\right) = \frac{1}{4}\left(10 - 7\sqrt{2}\right)$	M1
			$= \frac{1}{2\left(10 + 7\sqrt{2}\right)} > 0$	A1
			but both values are between $\frac{1}{2}\pi$ and π,	M1
			so both cosines are negative and hence $\cos^2(m(2)) > \cos^2(\mu(2)) \Rightarrow$ $0 > \cos(\mu(2)) > \cos(m(2))$	M1
			so $m(2) > \mu(2)$	A1
				[10]